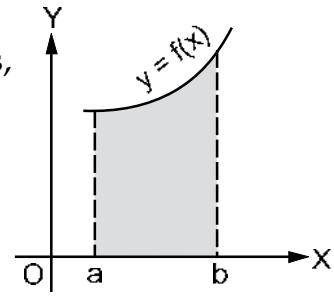
17. AREA OF BOUNDED REGIONS

1. AREA OF A CURVE BETWEEN TWO ORDINATES Let $y = f(x)$ be a continuous and finite function in $[a, b]$.

Case I *When the curve $y = f(x)$ lies above the x-axis*

The area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is given by

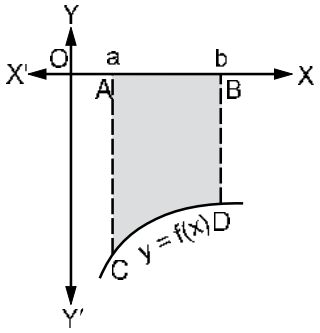
$$area = \int_a^b y\, dx.$$



Case II *When the curve $y = f(x)$ lies below the x-axis*

The area between the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$ is given by

$$area = \int_a^b (-y)\, dx.$$

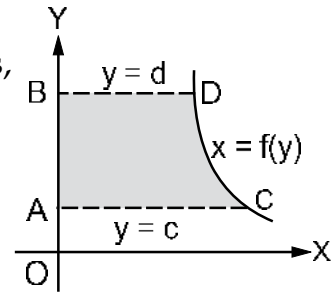


2. AREA OF A CURVE BETWEEN TWO ABSCISSAE

Case I *When the curve $x = f(y)$ lies to the right of the y-axis*

The area bounded by the curve $x = f(y)$, the y-axis, and the abscissae $y = c$ and $y = d$ is given by

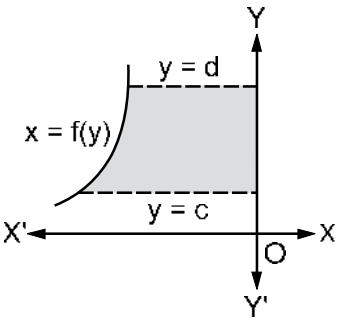
$$area = \int_c^d x\, dy.$$



Case II *When the curve $x = f(y)$ lies to the left of the y-axis*

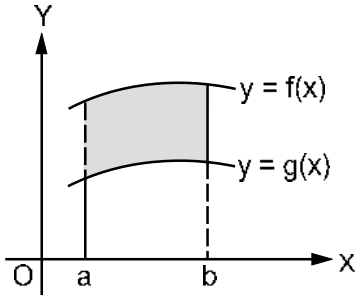
The area bounded by the curve $x = f(y)$, the y-axis, and the abscissae $y = c$ and $y = d$ is given by

$$area = \int_c^d (-x)\, dy.$$



3. AREA BETWEEN TWO CURVES The area bounded by two curves $y = f(x)$ and $y = g(x)$, which are intersected by the ordinates $x = a$ and $x = b$, is given by

$$area = \int_a^b \{f(x) - g(x)\}\, dx.$$

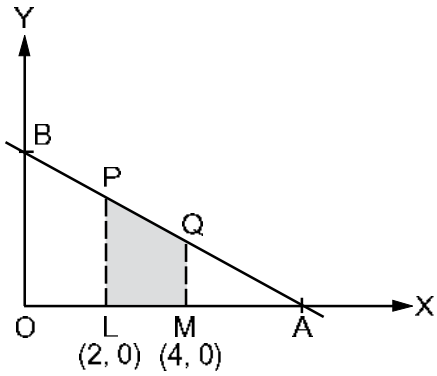


SOLVED EXAMPLES

EXAMPLE 1 *Using integration, find the area of the region bounded by the line* $2y + x = 8$, *the x-axis and the lines* $x = 2$ *and* $x = 4$. **[CBSE 2002C]**

SOLUTION The given line AB is

$$2y + x = 8 \Rightarrow y = 4 - \frac{1}{2}x. \quad \ldots \text{(i)}$$



Required area = area $PLMQP$

= area between the line $y = 4 - \frac{1}{2}x$,

and the x-axis between $x = 2$ and $x = 4$

$$= \int_2^4 y_{AB}\, dx = \int_2^4 \left(4 - \frac{1}{2}x\right) dx$$

$$= \left[4x - \frac{1}{4}x^2\right]_2^4 = (12 - 7) \text{ sq units}$$

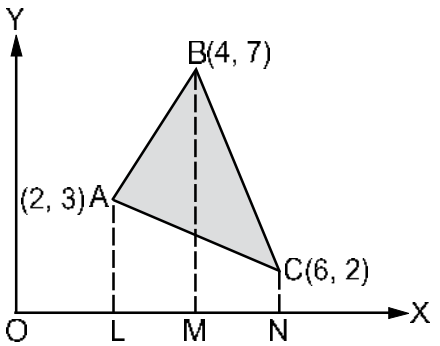
$= 5$ sq units.

Hence, the required area is 5 sq units.

EXAMPLE 2 *Using integration, find the area of* $\triangle ABC$ *whose vertices are* $A(2, 3)$, $B(4, 7)$ *and* $C(6, 2)$. **[CBSE 2001]**

SOLUTION The equation of AB is

$$\frac{y - 3}{x - 2} = \frac{7 - 3}{4 - 2} \Rightarrow y = 2x - 1.$$



The equation of BC is

$$\frac{y - 7}{x - 4} = \frac{2 - 7}{6 - 4} \Rightarrow y = \frac{-5}{2}x + 17.$$

The equation of AC is

$$\frac{y - 3}{x - 2} = \frac{2 - 3}{6 - 2} \Rightarrow y = \frac{-1}{4}x + \frac{7}{2}.$$

Draw $AL \perp OX$, $BM \perp OX$ and $CN \perp OX$.

Area of $\triangle ABC$ = (area $ALMBA$ + area $BMNCB$) − (area $ALNCA$)

$$= \int_2^4 y_{AB}\,dx + \int_4^6 y_{BC}\,dx - \int_2^6 y_{AC}\,dx$$

$$= \int_2^4 (2x-1)\,dx + \int_4^6 \left(-\frac{5}{2}x + 17\right)dx - \int_2^6 \left(-\frac{1}{4}x + \frac{7}{2}\right)dx$$

$$= [x^2 - x]_2^4 + \left[-\frac{5}{4}x^2 + 17x\right]_4^6 - \left[-\frac{1}{8}x^2 + \frac{7}{2}x\right]_2^6$$

$$= \left[(12-2) + (57-48) - \left(\frac{33}{2} - \frac{13}{2}\right)\right] \text{ sq units}$$

$$= 9 \text{ sq units.}$$

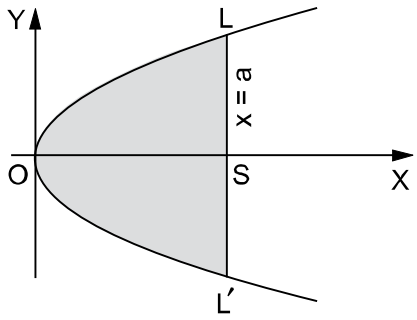
Hence, the required area is 9 sq units.

EXAMPLE 3 *Calculate the area bounded by the parabola $y^2 = 4ax$ and its latus rectum.*

SOLUTION Let $S(a, 0)$ be the focus of the parabola $y^2 = 4ax$. Then, its latus rectum LSL' is the line parallel to the y-axis at a distance a from it.

So, its equation is $x = a$.

Since the equation of the parabola contains only even powers of y, it is symmetrical about the x-axis.



$\therefore$ required area

$$= \text{area } LOL'L$$

$$= (\text{area } LOSL) + (\text{area } SOL'S)$$

$$= 2 \times (\text{area } LOSL)$$

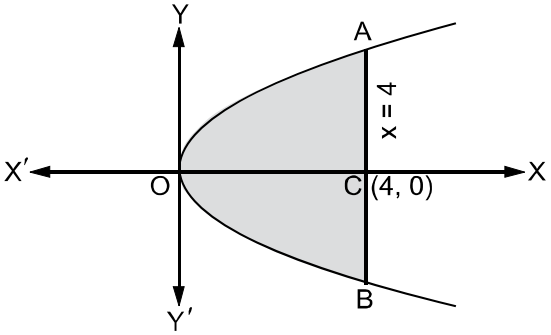
$$= 2\int_0^a y\,dx = 2 \cdot \int_0^a 2\sqrt{ax}\,dx$$

$$= 4\sqrt{a} \cdot \frac{2}{3}[x^{3/2}]_0^a = \frac{8}{3}a^2 \text{ sq units.}$$

EXAMPLE 4 *Using integration, find the area of the region bounded by the parabola $y^2 = 16x$ and the line $x = 4$.* **[CBSE 1996, '97]**

SOLUTION $y^2 = 16x$ is a right-handed parabola with its vertex at the origin.

And, $x = 4$ is the line parallel to the y-axis at a distance of 4 units from it.



Also, $y^2 = 16x$ contains only even powers of y.

So, it is symmetrical about the x-axis.

$\therefore$ required area = area $AOCA$ + area $BOCB$

$$= 2(\text{area } AOCA)$$

$$= 2\int_0^4 y\,dx = 2\int_0^4 \sqrt{16x}\,dx$$

$$= 8\int_0^4 \sqrt{x}\,dx = 8 \times \frac{2}{3} \times [x^{3/2}]_0^4$$

$$= \frac{16}{3} \times (4)^{3/2} = \left(\frac{16}{3} \times 8\right) = \frac{128}{3} \text{ sq units.}$$

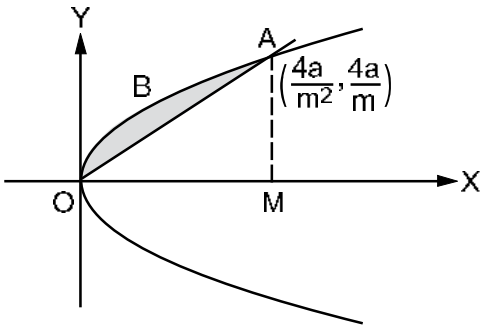
Hence, the required area is $\frac{128}{3}$ sq units.

EXAMPLE 5 *Using integration, find the area enclosed by the parabola $y^2 = 4ax$ and the chord $y = mx$.*

SOLUTION The given equations are

$$y^2 = 4ax \quad \ldots \text{(i)}$$

and $$y = mx \quad \ldots \text{(ii)}$$



Clearly, $y^2 = 4ax$ is a right-handed parabola, passing through the origin.

And $y = mx$ is a line passing through the origin.

In order to find the points of intersection of the given parabola and the given line, we solve (i) and (ii) simultaneously.

Putting $y = mx$ from (ii) into (i), we get

$$m^2x^2 = 4ax \Rightarrow x(m^2x - 4a) = 0$$

$$\Rightarrow x = 0 \text{ or } x = \frac{4a}{m^2}.$$

Now, $(x = 0 \Rightarrow y = 0)$ and $\left(x = \frac{4a}{m^2} \Rightarrow y = \frac{4a}{m}\right)$.

So, the points of intersection of the given parabola and the chord are

$$O(0, 0) \text{ and } A\left(\frac{4a}{m^2}, \frac{4a}{m}\right).$$

Draw $AM \perp OX$.

Required area = (area $OBAMO$) $-$ (area $OAMO$)

$$= \int_0^{4a/m^2} (y \text{ for the parabola})\, dx - \int_0^{4a/m^2} (y \text{ for the line})\, dx$$

$$= \int_0^{4a/m^2} 2\sqrt{ax}\, dx - \int_0^{4a/m^2} mx\, dx$$

$$= 2\sqrt{a} \cdot \frac{2}{3} [x^{3/2}]_0^{4a/m^2} - \left[\frac{mx^2}{2}\right]_0^{4a/m^2}$$

$$= \left[\frac{4\sqrt{a}}{3} \cdot \frac{8}{m^3} a^{3/2} - \frac{m}{2} \cdot \frac{16a^2}{m^4}\right]$$

$$= \left(\frac{32a^2}{3m^3} - \frac{8a^2}{m^3}\right) = \left(\frac{8a^2}{3m^3}\right) \text{ sq units.}$$

Hence, the required area is $\left(\frac{8a^2}{3m^3}\right)$ sq units.

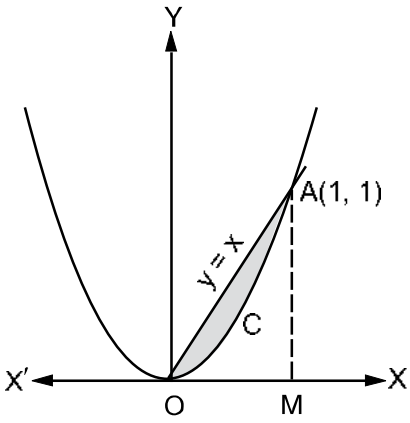
EXAMPLE 6 *Find the area of the region* $\{(x, y) : x^2 \le y \le x\}$. **[CBSE 2007, '13C]**

SOLUTION Consider the equations

$$y = x^2 \quad \ldots \text{(i)}$$

and $y = x \quad \ldots \text{(ii)}$

Clearly, $y = x^2$ is an upward parabola and $y = x$ is a line passing through (0, 0).



Solving (i) and (ii) simultaneously, we get

$$x^2 = x \Rightarrow x^2 - x = 0 \Rightarrow x(x-1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1.$$

From (ii), $(x = 0 \Rightarrow y = 0)$ and $(x = 1 \Rightarrow y = 1)$.

So, the points of intersection of (i) and (ii) are $O(0, 0)$ and $A(1, 1)$.

Draw $AM \perp OX$.

Required area = shaded area

$$= (\text{area } OMAO) - (\text{area } OMACO)$$

$$= \int_0^1 (y \text{ for line})\, dx - \int_0^1 (y \text{ for parabola})\, dx$$

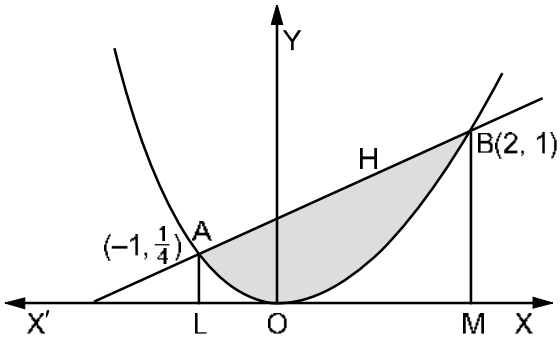
$$= \int_0^1 x\, dx - \int_0^1 x^2 dx = \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{6} \text{ sq unit.}$$

Hence, the required area is $\frac{1}{6}$ sq unit.

EXAMPLE 7 *Find the area of the region bounded by the curve* $x^2 = 4y$ *and the line* $x = 4y - 2$. **[CBSE 2010, '13]**

SOLUTION The given curve is $x^2 = 4y$ … (i)

The given line is $x = 4y - 2$ … (ii)



Putting $4y = (x + 2)$ from (ii) into (i), we get

$$x^2 = (x + 2) \Leftrightarrow (x^2 - x - 2) = 0$$

$$\Leftrightarrow (x - 2)(x + 1) = 0$$

$$\Leftrightarrow x = 2 \text{ or } x = -1.$$

Putting $x = 2$ in (i), we get $y = 1$.

Putting $x = -1$ in (i), we get $y = \frac{1}{4}$.

Thus, the points of intersection of the given curve (i) and the line (ii) are $A\left(-1, \frac{1}{4}\right)$ and $B(2, 1)$.

Draw AL and BM as perpendiculars on the x-axis.

$\therefore$ required area = area $AOBA$

$$= (\text{area } ALMBA) - (\text{area } AOBMLA)$$

$$= \int_{-1}^{2} \left(\frac{x+2}{4}\right) dx - \int_{-1}^{2} \frac{x^2}{4} dx$$

$$= \int_{-1}^{2} \left\{\frac{(x+2)}{4} - \frac{x^2}{4}\right\} dx = \frac{1}{4}\left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2}$$

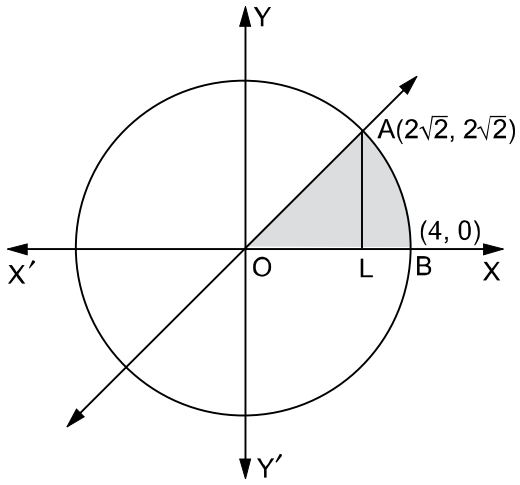
$$= \frac{1}{4}\left[\left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right)\right] = \frac{9}{8} \text{ sq units.}$$

Hence, the required area is $\frac{9}{8}$ sq units.

EXAMPLE 8 *Find the area bounded by the circle* $x^2 + y^2 = 16$ *and the line* $y = x$ *in the first quadrant.* **[CBSE 2004]**

SOLUTION The given circle is $x^2 + y^2 = 16$ … (i)

The given line is $y = x$ … (ii)



Putting $y = x$ from (ii) into (i), we get

$2x^2 = 16 \Leftrightarrow x^2 = 8 \Leftrightarrow x = 2\sqrt{2}$ [$\because$ x is + ve in the first quad.].

Thus, the point of intersection of (i) and (ii) in the first quadrant is $A(2\sqrt{2}, 2\sqrt{2})$.

Draw AL perpendicular on the x-axis.

$\therefore$ required area = (area OLA) + area($LBAL$)

$$= \int_0^{2\sqrt{2}} x\,dx + \int_{2\sqrt{2}}^{4} \sqrt{16 - x^2}\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^{2\sqrt{2}} + \left[\frac{x\sqrt{16-x^2}}{2} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_{2\sqrt{2}}^{4}$$

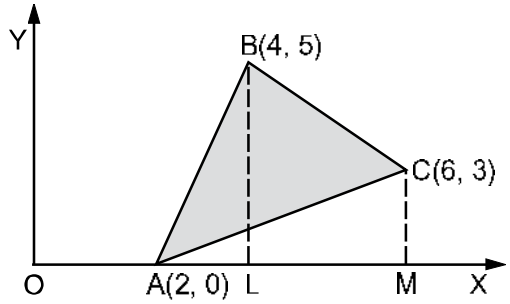
$$= \frac{1}{2}[(2\sqrt{2})^2 - 0] + \left[(0 + 8\sin^{-1}1) - (4 + 8\sin^{-1}\frac{1}{\sqrt{2}})\right]$$

$$= \left[4 + \left(8 \times \frac{\pi}{2}\right) - 4 - \left(8 \times \frac{\pi}{4}\right)\right] = (2\pi) \text{ sq units.}$$

EXAMPLE 9 *Using integration, find the area of $\triangle ABC$, whose vertices are $A(2, 0)$, $B(4, 5)$ and $C(6, 3)$.* **[CBSE 2003C, '06C]**

SOLUTION The equation of side AB is

$$\frac{y-0}{x-2} = \frac{(5-0)}{(4-2)} \Rightarrow y = \frac{5}{2}(x-2) \quad \ldots \text{(i)}$$



The equation of side BC is

$$\frac{y-5}{x-4}=\frac{(3-5)}{(6-4)} \Rightarrow y=-x+9 \qquad \ldots \text{(ii)}$$

The equation of side AC is

$$\frac{y-0}{x-2}=\frac{(3-0)}{(6-2)} \Rightarrow y=\frac{3}{4}(x-2) \qquad \ldots \text{(iii)}$$

Draw perpendiculars BL and CM on the x-axis.

$\therefore$ area of $\triangle ABC$

$$= \text{ar}(\triangle ALB) + \text{ar}(\text{trap. } BLMC) - \text{ar}(\triangle AMC)$$

$$= \int_2^4 y_{AB}\,dx + \int_4^6 y_{BC}\,dx - \int_2^6 y_{AC}\,dx$$

$$= \frac{5}{2}\int_2^4 (x-2)\,dx + \int_4^6 (9-x)\,dx - \frac{3}{4}\int_2^6 (x-2)\,dx$$

$$= \frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4 + \left[9x-\frac{x^2}{2}\right]_4^6 - \frac{3}{4}\cdot\left[\frac{x^2}{2}-2x\right]_2^6$$

$$= \frac{5}{2}[0-(-2)] + (36-28) - \frac{3}{4}[6-(-2)]$$

$$= (5+8-6) \text{ sq units} = 7 \text{ sq units.}$$

EXAMPLE 10 *Find the area cut off from the parabola* $4y = 3x^2$ *by the straight line* $3x - 2y + 12 = 0$. **[CBSE 2007, '09]**

SOLUTION Clearly, $4y = 3x^2$ is an upward parabola with its vertex at $(0, 0)$.

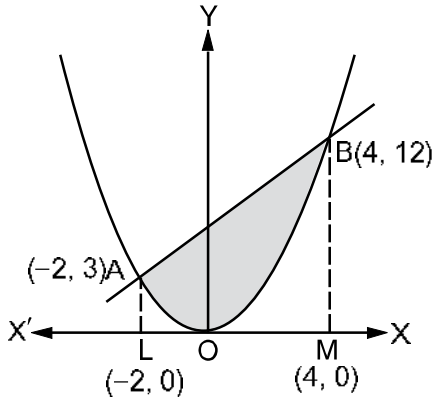
And, $3x - 2y + 12 = 0$ is a line.

The given equations are

$$4y = 3x^2 \qquad \ldots \text{(i)}$$

$$\text{and } 3x - 2y + 12 = 0 \qquad \ldots \text{(ii)}$$

The points of intersection of the given parabola and the given line will be obtained by solving (i) and (ii) simultaneously.



Putting $y = \frac{3}{4}x^2$ from (i) in (ii),

we get

$$3x - \frac{3}{2}x^2 + 12 = 0 \Rightarrow x^2 - 2x - 8 = 0$$

$$\Rightarrow x^2 - 4x + 2x - 8 = 0$$

$$\Rightarrow x(x-4) + 2(x-4) = 0$$

$$\Rightarrow (x-4)(x+2) = 0$$

$$\Rightarrow x = -2 \text{ or } x = 4.$$

Now, $(x = -2 \Rightarrow y = 3)$ and $(x = 4 \Rightarrow y = 12)$.

So, the points of intersection of (i) and (ii) are $A(-2, 3)$ and $B(4, 12)$.

Draw $AL \perp OX'$ and $BM \perp OX$.

Required area = (area $ALMBA$) − (area $ALOMBOA$)

$$= \int_{-2}^{4} (y \text{ of the line})\, dx - \int_{-2}^{4} (y \text{ of the parabola})\, dx$$

$$= \int_{-2}^{4} \frac{(3x+12)}{2}\, dx - \int_{-2}^{4} \frac{3x^2}{4}\, dx = \left[\frac{3x^2}{4} + 6x\right]_{-2}^{4} - \left[\frac{x^3}{4}\right]_{-2}^{4}$$

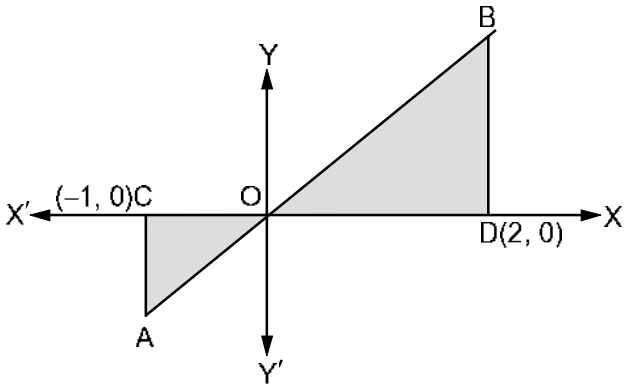
$$= (45 - 18) = 27 \text{ sq units.}$$

Hence, the required area is 27 sq units.

EXAMPLE 11 *Find the area bounded by the line $y = x$, the x-axis and the ordinates $x = -1$, $x = 2$.*

SOLUTION We know that $y = x$ is the line passing through the origin and making an angle of 45° with the x-axis, as shown in the given figure.

Now, we have to find the area of the shaded region.



Required area

$$= (\text{area } ODBO) + (\text{area } OACO)$$

$$= \int_{0}^{2} y\, dx + \int_{-1}^{0} (-y)\, dx \quad [\because \text{area } OACO \text{ is below the } x\text{-axis}]$$

$$= \int_{0}^{2} x\, dx + \int_{-1}^{0} (-x)\, dx$$

$$= \left[\frac{x^2}{2}\right]_{0}^{2} + \left[\frac{-x^2}{2}\right]_{-1}^{0} = \left[2 + \frac{1}{2}\right] = \frac{5}{2} \text{ sq units.}$$

Hence, the required area is $\frac{5}{2}$ sq units.

EXAMPLE 12 *Find by integration, the area of the region bounded by the curve* $y = 2x - x^2$ *and the x-axis.*

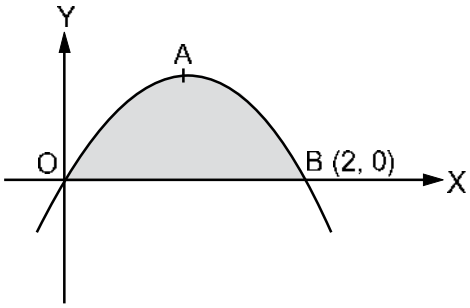
SOLUTION The given curve is $y = 2x - x^2$. … (i)

Now, $y = 2x - x^2 \Rightarrow (x^2 - 2x + 1) = (-y + 1)$

$\Rightarrow (x-1)^2 = -1(y-1)$

$\Rightarrow X^2 = -Y$,

where $(x-1) = X$ and $(y-1) = Y$.



Clearly, $X^2 = -Y$ is a downward parabola with its vertex at $(X = 0, Y = 0)$.

Now,

$X = 0, Y = 0 \Rightarrow x - 1 = 0$ and $y - 1 = 0$

$\Rightarrow x = 1$ and $y = 1$.

Thus, the vertex of the parabola is $A(1, 1)$.

Also, $y = 0 \Rightarrow 2x - x^2 = 0 \Rightarrow x(2-x) = 0 \Rightarrow x = 0$ or $x = 2$.

Thus, the curve cuts the x-axis at $O(0, 0)$ and $B(2, 0)$.

The rough sketch of the curve can now be drawn, as shown in the given figure.

$\therefore$ required area $= \int_0^2 y\,dx$

$= \int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{x^3}{3}\right]_0^2$

$= \left(4 - \frac{8}{3}\right) = \frac{4}{3}$ sq units.

Hence, the required area is $\frac{4}{3}$ sq units.

EXAMPLE 13 *Using integration, find the area of the region bounded by the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

SOLUTION The given equation contains only even powers of y.

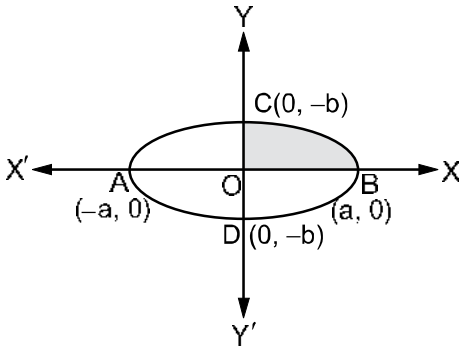
So, the curve is symmetrical about the x-axis.

Also, the given equation contains only even powers of x.

So, the curve is symmetrical about the y-axis.

A rough sketch of the ellipse can be drawn, as shown in the figure.

Now, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y=\frac{b}{a}\cdot\sqrt{a^2-x^2}$.



Area of the given ellipse

$= 4 \times (\text{area } OBCO)$

$= 4 \times \int_0^a y\,dx = 4 \times \int_0^a \frac{b}{a}\cdot\sqrt{a^2-x^2}\,dx$

$= \frac{4b}{a}\cdot\int_0^{\pi/2} a^2\cos^2\theta\,d\theta$ [putting $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$]

$= (4ab)\cdot\int_0^{\pi/2}\frac{(1+\cos 2\theta)}{2}\,d\theta = (2ab)\cdot\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

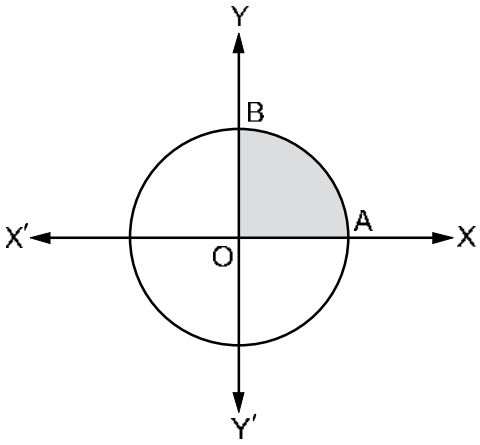
$= (\pi ab)$ sq units.

Hence, the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is (πab) sq units.

EXAMPLE 14 *By using integration, prove that the area of a circle of radius r units is πr^2 square units.*

SOLUTION The equation of a circle of radius r units with its centre at the origin is

$$x^2+y^2=r^2 \qquad \ldots \text{(i)}$$



This equation contains only even powers of y.

So, the curve is symmetrical about the x-axis.

Also, the above equation contains only even powers of x.

So, the curve is symmetrical about the y-axis.

Now, $x^2+y^2=r^2 \Rightarrow y=\sqrt{r^2-x^2}$.

Area of the circle $= 4 \times (\text{area } OABO)$

$= 4 \times \int_0^r y\,dx = 4 \times \int_0^r \sqrt{r^2-x^2}\,dx$

$= 4 \times \int_0^{\pi/2} r^2\cos^2\theta\,d\theta$ [putting $x = r\sin\theta$]

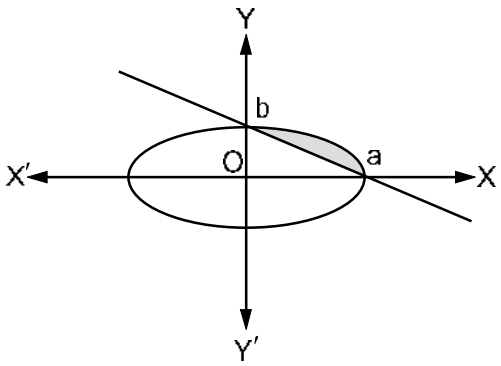
$= 4r^2\cdot\int_0^{\pi/2}\frac{(1+\cos 2\theta)}{2}\,d\theta = 2r^2\cdot\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$

$= (\pi r^2)$ sq units.

Hence, the required area is (πr^2) sq units.

EXAMPLE 15 *Find the area of the smaller region bounded by the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and the straight line* $\frac{x}{a}+\frac{y}{b}=1$. **[CBSE 2004, '05C]**

SOLUTION The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$ can be drawn, as shown in the given figure.



Then, we have to find the area of the shaded region.

Required area

$= \{\text{area between } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ and the } x\text{-axis from } x=0 \text{ to } x=a\}$

$\quad -\{\text{area between the line } \frac{x}{a}+\frac{y}{b}=1 \text{ and the } x\text{-axis from } x=0 \text{ to } x=a\}$

$= \int_0^a (y \text{ of the ellipse})\,dx - \int_0^a (y \text{ of the line})\,dx$

$= \int_0^a \frac{b}{a}\cdot\sqrt{a^2-x^2}\,dx - \int_0^a \frac{b(a-x)}{a}\,dx$

$= \frac{b}{a}\cdot\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a - \frac{b}{a}\cdot\left[ax-\frac{x^2}{2}\right]_0^a$

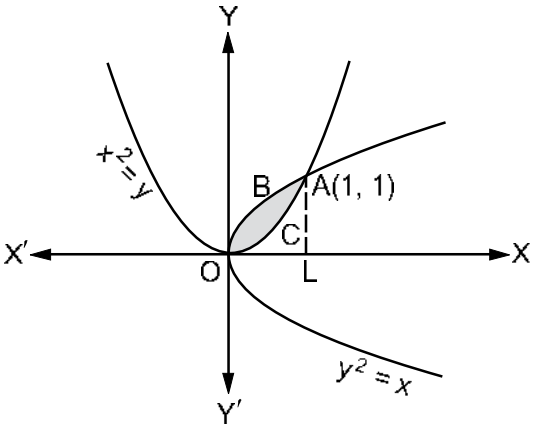
$= \frac{ab}{2}(\sin^{-1}1-\sin^{-1}0)-\left(ab-\frac{ab}{2}\right)=\left(\frac{\pi ab}{4}-\frac{ab}{2}\right)$ sq units.

Hence, the required area is $\left(\frac{\pi ab}{4}-\frac{ab}{2}\right)$ sq units.

EXAMPLE 16 *Find the area of the region bounded by the parabolas* $x^2=y$ *and* $y^2=x$. **[CBSE 2012C]**

SOLUTION Clearly, $x^2=y$ is an upward parabola with its vertex at (0, 0) and $y^2=x$ is a right-handed parabola with its vertex also at (0, 0).

The shaded region shows the area bounded by these parabolas.



The given equations are

$$x^2 = y \quad \ldots \text{(i)}$$

and $$y^2 = x \quad \ldots \text{(ii)}$$

Using (i) in (ii), we get

$$x^4 = x \Rightarrow x(x^3 - 1) = 0 \Rightarrow x = 0 \text{ or } x = 1.$$

Also, $(x = 0 \Rightarrow y = 0)$ and $(x = 1 \Rightarrow y = 1)$.

So, the given curves intersect at $O(0, 0)$ and $A(1, 1)$.

Draw $AL \perp OX$.

Required area = (area $OLABO$) − (area $OLACO$)

= {area bounded by $y^2 = x$ from $x = 0$ to $x = 1$}

− {area bounded by $x^2 = y$ from $x = 0$ to $x = 1$}

$$= \int_0^1 \sqrt{x}\, dx - \int_0^1 x^2\, dx$$

$$= \left[\frac{2}{3} x^{3/2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1 = \left(\frac{2}{3} - \frac{1}{3}\right) = \frac{1}{3} \text{ sq unit.}$$

Hence, the required area is $\frac{1}{3}$ sq unit.

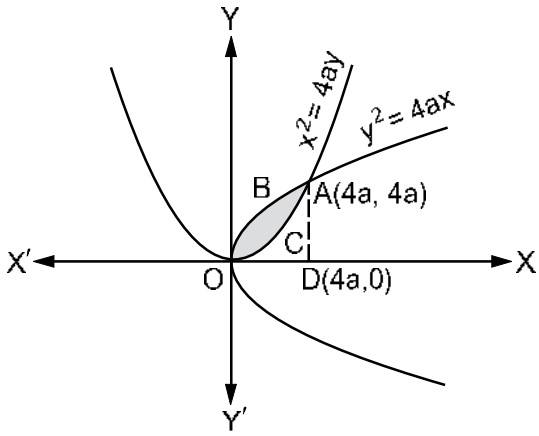
EXAMPLE 17 *Find the area of the region included between the parabolas* $y^2 = 4ax$ *and* $x^2 = 4ay$, *where* $a > 0$.

[CBSE 2003, '04, '08, '09, '12C]

SOLUTION The given parabolas are

$$y^2 = 4ax \quad \ldots \text{(i)}$$

and $$x^2 = 4ay \quad \ldots \text{(ii)}$$



In order to find the points of intersection of the given curves, we solve (i) and (ii) simultaneously.

Putting $x = \frac{y^2}{4a}$ from (i) in (ii), we get

$$\frac{y^4}{16a^2} = 4ay \Rightarrow y^4 - 64a^3 y = 0$$

$$\Rightarrow y(y^3 - 64a^3) = 0$$

$$\Rightarrow y = 0 \text{ or } y = 4a.$$

Now, $(y = 0 \Rightarrow x = 0)$ and $(y = 4a \Rightarrow x = \frac{16a^2}{4a} = 4a)$.

Thus, the points of intersection of the two parabolas are $O(0, 0)$ and $A(4a, 4a)$.

Draw $AD \perp OX$. Then, point D is $(4a, 0)$.

Required area = area $OCABO$

$$= (\text{area } OBADO) - (\text{area } OCADO)$$

$$= \int_0^{4a} y\,dx \text{ for } (y^2 = 4ax) - \int_0^{4a} y\,dx \text{ for } (x^2 = 4ay)$$

$$= \int_0^{4a} 2\sqrt{ax}\,dx - \int_0^{4a} \frac{x^2}{4a}dx$$

$$= \left[2\sqrt{a}\cdot\frac{2}{3}\cdot x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

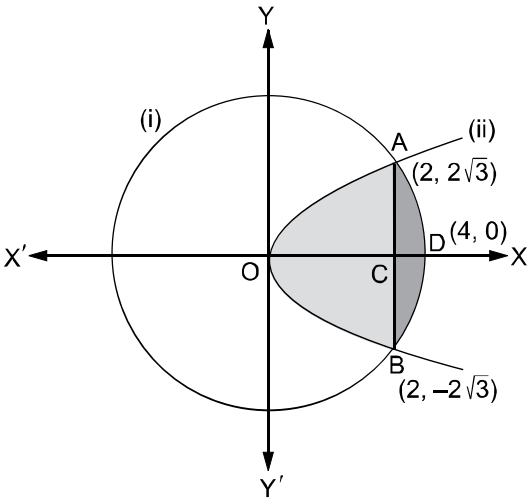
$$= \left[\frac{4\sqrt{a}}{3}\cdot(4a)^{3/2} - \frac{1}{12a}\times 64a^3\right]$$

$$= \left(\frac{32a^2}{3} - \frac{16a^2}{3}\right) = \left(\frac{16a^2}{3}\right) \text{ sq units.}$$

Hence, the required area is $\left(\frac{16a^2}{3}\right)$ sq units.

EXAMPLE 18 *Using integration, find the area of the region common to the circle* $x^2 + y^2 = 16$ *and the parabola* $y^2 = 6x$. **[CBSE 2012C]**

SOLUTION $x^2 + y^2 = 16$ is a circle with its centre at (0, 0) and radius = 4 units. And, $y^2 = 6x$ is a right-handed parabola with its vertex at (0, 0).



Now, we have to find the area of the shaded region.

The given equations are

$$x^2 + y^2 = 16 \quad \ldots \text{(i)}$$

and $$y^2 = 6x \quad \ldots \text{(ii)}$$

Using (ii) in (i), we get

$$x^2 + 6x - 16 = 0 \Rightarrow (x+8)(x-2) = 0$$

$$\Rightarrow x = -8 \text{ or } x = 2.$$

Now, $x = -8 \Rightarrow y^2 = -48$

$\Rightarrow y$ is imaginary.

And, $x = 2 \Rightarrow y^2 = (6\times 2) = 12$

$\Rightarrow y = \pm 2\sqrt{3}$.

Thus, the points of intersection of the given curves are $A(2, 2\sqrt{3})$ and $B(2, -2\sqrt{3})$.

Since each of the given equations contains only even powers of y, each one is symmetrical about the x-axis.

$\therefore$ required area $= 2(\text{area } OCDAO)$

$= 2(\text{area } OCAO + \text{area } CDAC)$

$= 2\left[\int_0^2 y\,dx \text{ for curve (ii)} + \int_2^4 y\,dx \text{ for curve (i)}\right]$

$= 2\left[\int_0^2 \sqrt{6x}\,dx + \int_2^4 \sqrt{16 - x^2}\,dx\right]$

$= 2\left\{\left[\frac{2\sqrt{6}}{3}\,x^{3/2}\right]_0^2 + \left[x\sqrt{\frac{16-x^2}{2}} + \frac{16}{2}\sin^{-1}\frac{x}{4}\right]_2^4\right\}$

$= \left\{\left(\frac{2\sqrt{6}}{3}\cdot 2^{3/2} - 0\right) + 8\sin^{-1}1 - \left(2\sqrt{3} + 8\sin^{-1}\frac{1}{2}\right)\right\}$

$= 2\left(\frac{8\sqrt{3}}{3} + 4\pi - 2\sqrt{3} - \frac{4\pi}{3}\right) = 2\left(\frac{2\sqrt{3}}{3} + \frac{8\pi}{3}\right)$

$= \frac{4}{3}(\sqrt{3} + 4\pi)$ sq units.

Hence, the required area is $\frac{4}{3}(\sqrt{3} + 4\pi)$ sq units.

EXAMPLE 19 *Using integration, find the area of the region enclosed between the circles* $x^2 + y^2 = 4$ *and* $(x-2)^2 + y^2 = 4$. **[CBSE 2009, '13C]**

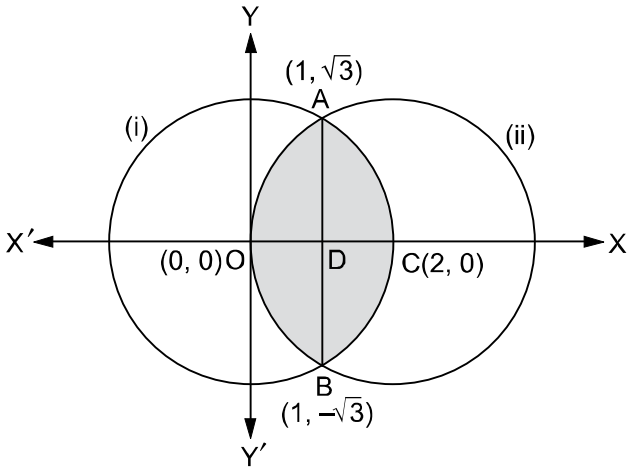
SOLUTION $x^2 + y^2 = 4$ is a circle with its centre at $O(0, 0)$ and radius = 2 units.

And, $(x-2)^2 + y^2 = 4$ is a circle with its centre at $C(2, 0)$ and radius = 2 units.

The given circles are

$x^2 + y^2 = 4$ … (i)

and $(x-2)^2 + y^2 = 4$ … (ii)



Eliminating y from (i) and (ii), we get

$$4-x^2=4-(x-2)^2 \Rightarrow 4x=4 \Rightarrow x=1.$$

Putting $x=1$ in (i), we get $y^2=3 \Rightarrow y=\pm\sqrt{3}$.

Thus, the points of intersection of the two circles are $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$.

Both the circles are symmetrical about the x-axis.

Required area $= 2$(area $AOCA$)

$= 2$(area $AODA$ + area $CADC$)

$$= 2\int_0^1 y\,dx \text{ for circle (ii)} + 2\int_1^2 y\,dx \text{ for circle (i)}$$

$$= 2\int_0^1 \sqrt{4-(x-2)^2}\,dx + 2\int_1^2 \sqrt{4-x^2}\,dx$$

$$= \left[\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\cdot\sin^{-1}\frac{(x-2)}{2}\right]_0^1 + \left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\cdot\sin^{-1}\frac{x}{2}\right]_1^2$$

$$= 2\left[\left\{\frac{-\sqrt{3}}{2} + 2\sin^{-1}\left(\frac{-1}{2}\right)\right\} - \{0 + 2\sin^{-1}(-1)\} + \left\{2\sin^{-1}(1) - \frac{\sqrt{3}}{2} - 2\sin^{-1}\left(\frac{1}{2}\right)\right\}\right]$$

$$= 2\left\{\frac{-\sqrt{3}}{2} + 2\left(-\frac{\pi}{6}\right) - 2\left(-\frac{\pi}{2}\right) + \left(2\cdot\frac{\pi}{2} - \frac{\sqrt{3}}{2} - 2\times\frac{\pi}{6}\right)\right\}$$

$$= 2\left(\frac{4\pi}{3} - \sqrt{3}\right) \text{ sq units.}$$

Hence, the required area is $2\left(\frac{4\pi}{3} - \sqrt{3}\right)$ sq units.

EXAMPLE 20 *Using integration, find the area of the region*

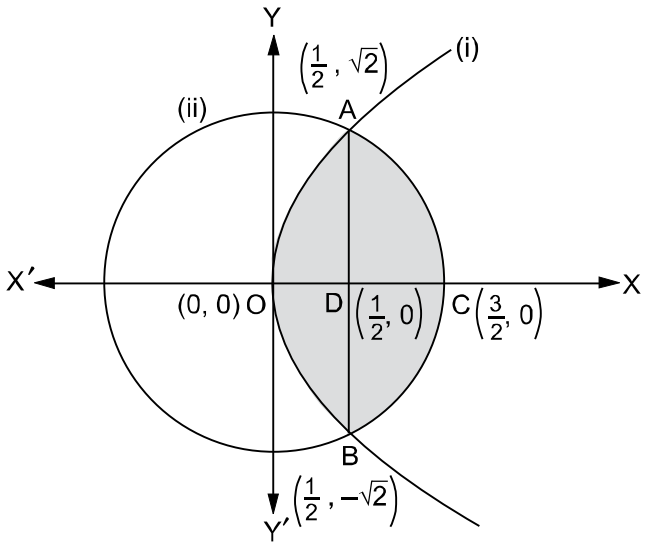
$$\{(x, y) : y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\}.$$

[CBSE 2008, '13]

SOLUTION Clearly, we have to find the area enclosed between the curves $y^2 = 4x$ and $4x^2 + 4y^2 = 9$.

The curve $y^2 = 4x$ is a right-handed parabola with its vertex at (0, 0).

$4x^2 + 4y^2 = 9 \Rightarrow x^2 + y^2 = \left(\frac{3}{2}\right)^2$, which represents a circle with its centre at $O(0, 0)$ and radius equal to $(3/2)$ units.



The shaded region shows the required area.

Now, $y^2 = 4x$ … (i)

and $4x^2 + 4y^2 = 9$ … (ii)

Putting the value of y^2 from (i) into (ii), we get

$4x^2 + 16x - 9 = 0$

$\Rightarrow \; 4x^2 + 18x - 2x - 9 = 0$

$\Rightarrow \; 2x(2x + 9) - (2x + 9) = 0$

$\Rightarrow \; (2x + 9)(2x - 1) = 0$

$\Rightarrow \; x = -\frac{9}{2}$ or $x = \frac{1}{2}$.

Putting $x = -\frac{9}{2}$ in (i), we get

$y^2 = -18 \Rightarrow y$ is imaginary.

Putting $x = \frac{1}{2}$ in (i), we get $y^2 = 2 \Rightarrow y = \pm\sqrt{2}$.

So, the given curves intersect at the points $A\left(\frac{1}{2}, \sqrt{2}\right)$ and $B\left(\frac{1}{2}, -\sqrt{2}\right)$.

Since the equation of each of the given curves contains only even powers of y, each curve is symmetrical about the x-axis.

Required area $= 2$ (area $ODCAO$)

$= 2$ (area $ODAO$ + area $DCAD$)

$= 2\left\{\int_0^{1/2} (y \text{ of the parabola})\, dx + \int_{1/2}^{3/2} (y \text{ of the circle})\, dx\right\}$

$$= 2\left\{\int_0^{1/2} 2\sqrt{x}\,dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4} - x^2}\,dx\right\}$$

$$= 2\left\{\left[\frac{4}{3}x^{3/2}\right]_0^{1/2} + \left[\frac{x\sqrt{\frac{9}{4}-x^2}}{2} + \frac{9}{8}\sin^{-1}\left(\frac{x}{3/2}\right)\right]_{1/2}^{3/2}\right\}$$

$$= \frac{4}{3\sqrt{2}} + \left(\frac{9}{4}\sin^{-1}1\right) - \left(\frac{\sqrt{2}}{2} + \frac{9}{4}\sin^{-1}\frac{1}{3}\right)$$

$$= \left(\frac{2\sqrt{2}}{3} - \frac{\sqrt{2}}{2}\right) + \frac{9}{4}\left(\sin^{-1}1 - \sin^{-1}\frac{1}{3}\right)$$

$$= \left\{\frac{\sqrt{2}}{6} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right)\right\}\text{sq units}$$

$$= \left\{\frac{\sqrt{2}}{6} + \frac{9}{4}\cos^{-1}\left(\frac{1}{3}\right)\right\}\text{sq units.}$$

Hence, the required area is $\left\{\frac{\sqrt{2}}{6} + \frac{9}{4}\cos^{-1}\frac{1}{3}\right\}$ sq units.

EXAMPLE 21 *Find the area of the region* $\{(x, y) : x^2 + y^2 \le 1 \le x + y\}$.

SOLUTION Let $R = \{(x, y) : x^2 + y^2 \le 1 \le x + y\}$

$= \{(x, y) : x^2 + y^2 \le 1\} \cap \{(x, y) : x + y \ge 1\}$

$= R_1 \cap R_2$.

Clearly, R_1 is the interior of the circle $x^2 + y^2 = 1$ with its centre at $O(0, 0)$ and radius = 1 unit.

And, R_2 is the region lying above the line $x + y = 1$.

Consider the equations

$$x^2 + y^2 = 1 \qquad \ldots \text{(i)}$$

and $$x + y = 1 \qquad \ldots \text{(ii)}$$

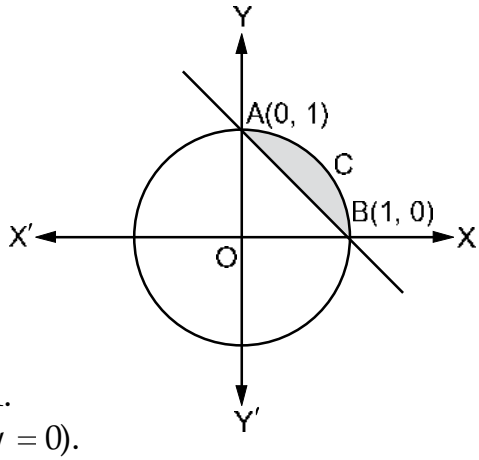
Putting $y = (1 - x)$ from (ii) in (i), we get

$$x^2 + (1 - x)^2 = 1 \Rightarrow 2x^2 - 2x = 0$$

$$\Rightarrow 2x(x - 1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1.$$

Now, $(x = 0 \Rightarrow y = 1)$ and $(x = 1 \Rightarrow y = 0)$.



Thus, the points of intersection of (i) and (ii) are $A(0, 1)$ and $B(1, 0)$.

So, the required area is the shaded region.

Required area $=$ area $BCAB$

$= (\text{area } AOBCA) - (\text{area } OBAO)$

$= \int_0^1 \sqrt{1-x^2}\, dx - \int_0^1 (1-x)\, dx$

$= \left[\frac{1}{2}\sin^{-1} x + \frac{x}{2}\sqrt{1-x^2}\right]_0^1 - \left[x - \frac{x^2}{2}\right]_0^1$

$= \left(\frac{1}{2}\sin^{-1} 1\right) - \frac{1}{2} = \left(\frac{1}{2} \times \frac{\pi}{2}\right) - \frac{1}{2} = \left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units.

Hence, the required area is $\left(\frac{\pi}{4} - \frac{1}{2}\right)$ sq units.

EXAMPLE 22 *Find the area of the region* $\{(x, y) : x^2 + y^2 \le 2ax,\ y^2 \ge ax,\ x \ge 0,\ y \ge 0\}$.

SOLUTION Clearly, we have to find the area of the region lying in the first quadrant ($x \ge 0,\ y \ge 0$), included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.

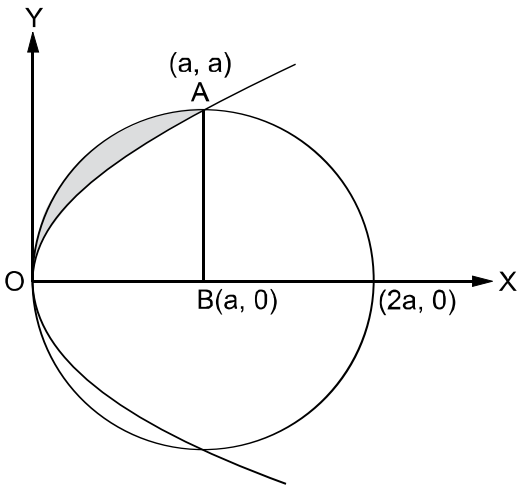
Thus, the equations of the given curves are

$$x^2 + y^2 = 2ax \quad \dots \text{(i)}$$

and $$y^2 = ax \quad \dots \text{(ii)}$$

Now, clearly $x^2 + y^2 = 2ax$ is a circle with its centre $B(a, 0)$ and radius $= a$ units.

And, $y^2 = ax$ is a parabola with $O(0, 0)$ as its vertex and the x-axis as its axis.



We can draw the figure, as shown.

Their points of intersection may be obtained by solving (i) and (ii) and keeping in view that $x \ge 0$ and $y \ge 0$.

Using (ii) in (i), we get

$$x^2 - ax = 0 \Rightarrow x(x-a) = 0$$
$$\Rightarrow x = 0 \text{ or } x = a.$$

Now, $(x = 0 \Rightarrow y = 0)$ and $(x = a \Rightarrow y = a)$.

Thus, the two curves intersect at $O(0, 0)$ and $A(a, a)$.

$\therefore$ required area $= \int_0^a \sqrt{2ax - x^2}\, dx - \int_0^a \sqrt{ax}\, dx$

$= \int_0^a \sqrt{a^2 - (x-a)^2}\, dx - \sqrt{a} \cdot \int_0^a \sqrt{x}\, dx$

$$= \left[\frac{(x-a)\sqrt{a^2-(x-a)^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x-a}{a}\right) \right]_0^a$$

$$-\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^a$$

$$= \left\{ \frac{a^2}{2}\sin^{-1}(0) - \frac{a^2}{2}\sin^{-1}(-1) - \frac{2}{3}a^2 \right\}$$

$$= \left(\frac{\pi a^2}{4} - \frac{2}{3}a^2 \right) \text{ sq units.}$$

Hence, the required area is $\left(\frac{\pi a^2}{4} - \frac{2}{3}a^2 \right)$ sq units.

EXAMPLE 23 *Find the area of the region* $\{(x, y) : x^2 \le y \le |x|\}$. **[CBSE 2009, '12C, '13]**

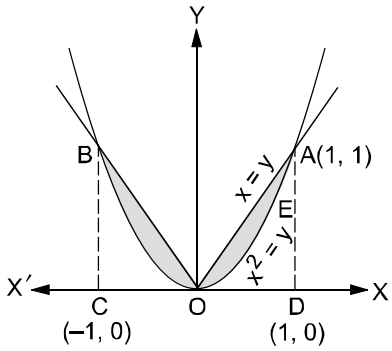
SOLUTION Consider the equations

$$x^2 = y \quad \ldots \text{(i)}$$

and $$y = |x| \quad \ldots \text{(ii)}$$

Clearly, $x^2 = y$ represents an upward parabola with its vertex at $O(0, 0)$. All the points inside this parabola represent $x^2 \le y$.

Also, $y = |x| = \begin{cases} x, \text{ when } x \ge 0 \\ -x, \text{ when } x < 0. \end{cases}$



The lines OA and OB, each equally inclined to the axes, represent $y = |x|$.

All the points below the lines OA and OB, and above the x-axis represent $y \le |x|$.

Thus, the shaded portion is the required region.

In each of the given equations, the equation remains unchanged when x is replaced by $-x$.

So, each of the given curves is symmetrical about the y-axis.

$\therefore$ required area $= 2$(area $OEAO$).

In this region, we have

$$x^2 = y \quad \ldots \text{(iii)}$$

and $$y = x \quad \ldots \text{(iv)}$$

Using (iv) in (iii), we get $x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$.

Now, $(x = 0 \Rightarrow y = 0)$ and $(x = 1 \Rightarrow y = 1)$.

Thus, the line $y = x$ and the curve $x^2 = y$ intersect at $O(0, 0)$ and $A(1, 1)$. Draw $AD \perp OX$ and $BC \perp OX'$.

$\therefore$ required area

$= 2(\text{area } OEAO) = 2[(\text{area } ODAO) - (\text{area } ODAEO)]$

$= 2[(\text{ area between } y = x \text{ and the } x\text{-axis from } x = 0 \text{ to } x = 1)$

$- (\text{area between } x^2 = y \text{ and the } x\text{-axis from } x = 0 \text{ to } x = 1)]$

$$= 2\left(\int_0^1 y\,dx \text{ for the line } OA - \int_0^1 y\,dx \text{ for the curve } OEA\right)$$

$$= 2\left[\int_0^1 x\,dx - \int_0^1 x^2\,dx\right] = 2\left\{\left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^3}{3}\right]_0^1\right\} = 2\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$= \left(2 \times \frac{1}{6}\right) = \frac{1}{3} \text{ sq unit.}$$

Hence, the required area is $\frac{1}{3}$ sq unit.

EXAMPLE 24 *Find the area bounded by the line $y = x$ and the curve $y = x^3$.*

SOLUTION The given equations are

$$y = x \quad \ldots \text{(i)}$$

and $$y = x^3 \quad \ldots \text{(ii)}$$

Using (i) in (ii), we get

$$x - x^3 = 0 \Rightarrow x(1 - x^2) = 0 \Rightarrow x(1 - x)(1 + x) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1 \text{ or } x = -1.$$

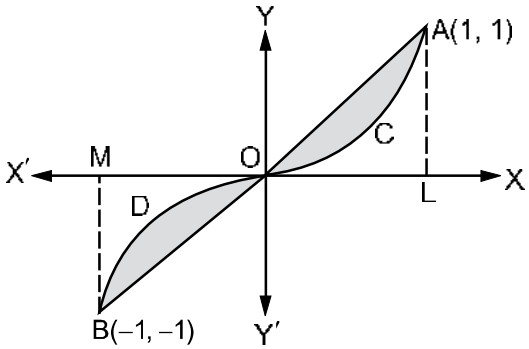
Also, $(x = 0 \Rightarrow y = 0)$, $(x = 1 \Rightarrow y = 1)$ and $(x = -1 \Rightarrow y = -1)$.

So, the given curve and the line intersect at the points $O(0, 0)$, $A(1, 1)$ and $B(-1, -1)$.

Now, $y = x$ is a line passing through the origin and making an angle of $45°$ with the x-axis. Thus, the line $y = x$ can be drawn.

For the curve $y = x^3$ some values for x and the corresponding values of y are given below:

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y	-1	$-\frac{1}{8}$	0	$\frac{1}{8}$	1



Plotting the points $(-1, -1)$, $\left(-\frac{1}{2}, -\frac{1}{8}\right)$, $(0, 0)$, $\left(\frac{1}{2}, \frac{1}{8}\right)$, and $(1, 1)$, and joining them, we get a rough sketch of $y = x^3$, as shown in the given figure.

Required area

$$= (\text{area } ACOA) + (\text{area } ODBO)$$

$$= (\text{area } OALO) - (\text{area } OCALO) + (\text{area } OBMO) - (\text{area } ODBMO)$$

$$= \int_0^1 \{y \text{ for (i)}\}\, dx - \int_0^1 \{y \text{ for (ii)}\}\, dx + \int_{-1}^0 \{(-y) \text{ for (i)}\}\, dx - \int_{-1}^0 \{(-y) \text{ for (ii)}\}\, dx$$

$$= \int_0^1 x\, dx - \int_0^1 x^3\, dx + \int_{-1}^0 -x\, dx - \int_{-1}^0 -x^3\, dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1 + \left[\frac{-x^2}{2}\right]_{-1}^0 + \left[\frac{x^4}{4}\right]_{-1}^0$$

$$= \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2} \text{ sq unit.}$$

Hence, the required area is 0.5 sq unit.

EXAMPLE 25 *Find the area bounded by the curve* $y = \sin x$ *between* $x = 0$ *and* $x = 2\pi$.

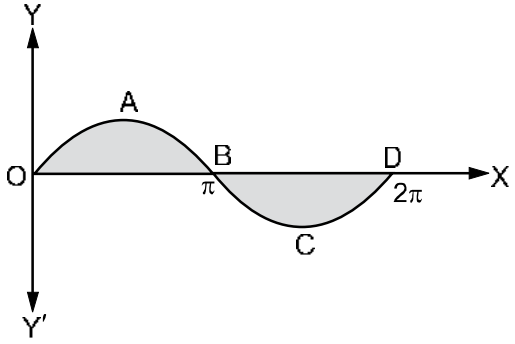
SOLUTION The given curve is $y = \sin x$.

Some values of x and the corresponding values of y are given below:

x	0	$\frac{\pi}{6}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	$\frac{1}{2}$	1	0	-1	0

Taking a fixed unit (distance) for π along the x-axis, we can plot the points $(0, 0)$, $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\left(\frac{\pi}{2}, 1\right)$, $(\pi, 0)$, $\left(\frac{3\pi}{2}, -1\right)$ and $(2\pi, 0)$.

Join these points freehand to obtain a rough sketch of the given curve.



Required area = (area $OABO$) + (area $BCDB$)

$$= \int_0^{\pi} y\,dx + \int_{\pi}^{2\pi} (-y)\,dx \quad [\because \text{area } BCDB \text{ is below the } x\text{-axis}]$$

$$= \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx$$

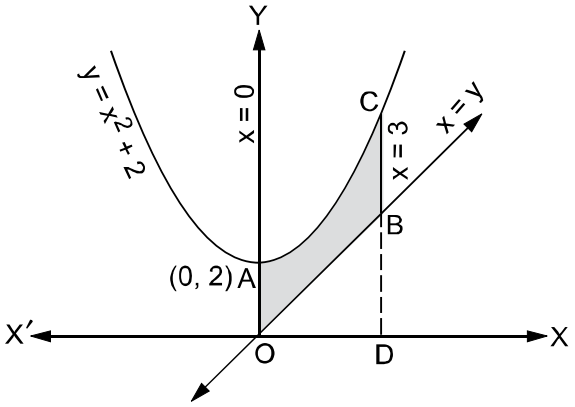
$$= [-\cos x]_0^{\pi} - [-\cos x]_{\pi}^{2\pi} = (2+2) = 4 \text{ sq units.}$$

Hence, the required area is 4 sq units.

EXAMPLE 26 *Find the area of the region bounded by the curve* $y = x^2 + 2$, *and the lines* $y = x$, $x = 0$ *and* $x = 3$.

SOLUTION $y = x^2 + 2 \Rightarrow x^2 = (y - 2)$.

Clearly, $x^2 = (y-2)$ represents an upward parabola with its vertex at $A(0, 2)$.



Also, $y = x$ represents the straight line, making an angle of $45°$ with the positive direction of the x-axis.

And, $x = 0$ is the y-axis, while $x = 3$ represents a line parallel to the y-axis at a distance of 3 units from it.

Thus, the shaded region in the given figure is the required area.

$\therefore$ required area = (area $ODCAO$) − (area $ODBO$)

$$= \int_0^3 (x^2 + 2)\,dx - \int_0^3 x\,dx$$

$$= \left[\frac{x^3}{3} + 2x\right]_0^3 - \left[\frac{x^2}{2}\right]_0^3 = \left(15 - \frac{9}{2}\right) = \frac{21}{2} \text{ sq units.}$$

Hence, the required area is $\frac{21}{2}$ sq units.

EXAMPLE 27 *Find the area of the region*

$\{(x, y) : 0 \le y \le (x^2 + 1), 0 \le y \le (x + 1), 0 \le x \le 2\}$. **[CBSE 2009]**

SOLUTION Let $R = \{(x, y) : 0 \le y \le (x^2 + 1),\ 0 \le y \le (x + 1),\ 0 \le x \le 2\}$

$= \{(x, y) : 0 \le y \le (x^2 + 1)\} \cap \{(x, y) : 0 \le y \le (x + 1)\}$

$\cap \{(x, y) : 0 \le x \le 2\}$

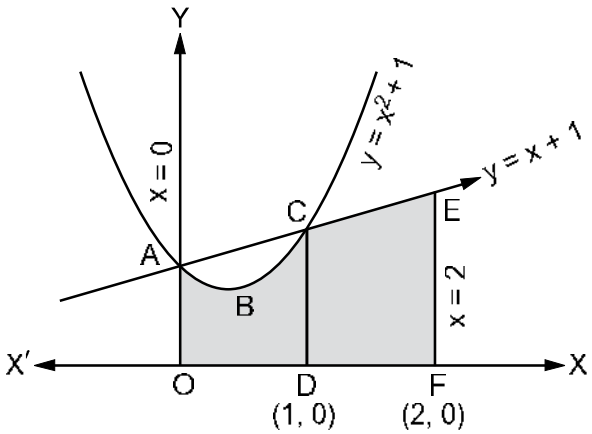
$= R_1 \cap R_2 \cap R_3.$

Clearly, R_1 is the region consisting of the right-hand side of the y-axis, lying below the parabola $y = x^2 + 1$.

Also, R_2 is the region consisting of the right-hand side of the y-axis, lying below the line $y = (x + 1)$.

And, R_3 is the region above the x-axis, lying between the ordinates $x = 0$ and $x = 2$.

Thus, $R_1 \cap R_2 \cap R_3$ is the shaded region.



We have, $y = x^2 + 1$ and $y = x + 1$

$\Rightarrow x^2 + 1 = x + 1 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0$ or $x = 1$.

Now, $(x = 0 \Rightarrow y = 1)$ and $(x = 1 \Rightarrow y = 2)$.

Thus, the parabola $y = (x^2 + 1)$ and the line $y = x + 1$ intersect at the points $A(0, 1)$ and $C(1, 2)$.

$\therefore$ required area = area of the shaded region

$= (\text{area } ODCBA) + (\text{area } CDFEC)$

$= \int_0^1 (y \text{ of the parabola})\, dx + \int_1^2 (y \text{ of the line})\, dx$

$= \int_0^1 (x^2 + 1)\, dx + \int_1^2 (x + 1)\, dx$

$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2$

$= \left(\frac{1}{3} + 1\right) + \left(4 - \frac{3}{2}\right) = \frac{23}{6}$ sq units.

Hence, the required area is $\frac{23}{6}$ sq units.

EXAMPLE 28 *Find the area of the region bounded by the curve* $y^2 = 2y - x$ *and the y-axis.*

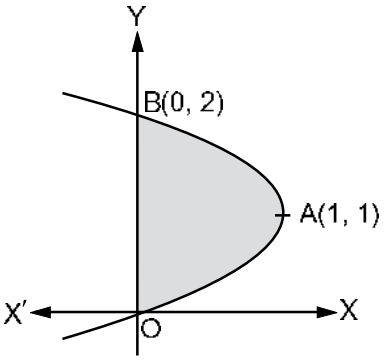
SOLUTION $y^2 = 2y - x \Rightarrow y^2 - 2y = -x$

$\Rightarrow y^2 - 2y + 1 = -x + 1$

$\Rightarrow (y-1)^2 = -(x-1)$

$\Rightarrow Y^2 = -X,$

where $y - 1 = Y$ and $(x - 1) = X$.



This is a left-handed parabola with vertex at $(X = 0, Y = 0)$.

$X = 0, Y = 0 \Rightarrow -x + 1 = 0$ and $y - 1 = 0$

$\Rightarrow x = 1$ and $y = 1$.

Thus, the vertex of the given parabola is $A(1, 1)$.

Also, $x = 0 \Rightarrow y^2 - 2y = 0 \Rightarrow y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$.

Thus, the curve meets the y-axis at $O(0, 0)$ and $B(0, 2)$.

A rough sketch of the curve can be drawn, as shown in the figure.

$\therefore$ required area $= \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy$

$$= \left[y^2 - \frac{y^3}{3} \right]_0^2 = \left(4 - \frac{8}{3} \right) = \frac{4}{3} \text{ sq units.}$$

Hence, the required area is $\frac{4}{3}$ sq units.

EXERCISE 17

1. Find the area of the region bounded by the curve $y = x^2$, the x-axis, and the lines $x = 1$ and $x = 3$.
2. Find the area of the region bounded by the parabola $y^2 = 4x$, the x-axis, and the lines $x = 1$ and $x = 4$.
3. Find the area under the curve $y = \sqrt{6x + 4}$ (above the x-axis) from $x = 0$ to $x = 2$.
4. Determine the area enclosed by the curve $y = x^3$, and the lines $y = 0$, $x = 2$ and $x = 4$.
5. Determine the area under the curve $y = \sqrt{a^2 - x^2}$, included between the lines $x = 0$ and $x = 4$.
6. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, the x-axis, and the lines $x = 2$ and $x = 8$.
7. Find the area of the region bounded by the curve $y^2 = 4x$ and the line $x = 3$.

8. Evaluate the area bounded by the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ above the x-axis.

9. Using integration, find the area of the region bounded by the lines $y=1+|x+1|$, $x=-2$, $x=3$ and $y=0$.

10. Find the area bounded by the curve $y=(4-x^2)$, the y-axis and the lines $y=0$, $y=3$.

11. Using integration, find the area of the region bounded by the triangle whose vertices are $A(-1, 2)$, $B(1, 5)$ and $C(3, 4)$. **[CBSE 2014]**

12. Using integration, find the area of the $\triangle ABC$, the equations of whose sides AB, BC and AC are given by
$y=4x+5$, $x+y=5$ and $4y=x+5$ respectively.

13. Using integration, find the area of the region bounded between the line $x=2$ and the parabola $y^2=8x$.

14. Using integration, find the area of the region bounded by the line $y-1=x$, the x-axis, and the ordinates $x=-2$ and $x=3$.

15. Sketch the region lying in the first quadrant and bounded by $y=4x^2$, $x=0$, $y=2$ and $y=4$. Find the area of the region using integration.

16. Sketch the region lying in the first quadrant and bounded by $y=9x^2$, $x=0$, $y=1$ and $y=4$. Find the area of the region, using integration.

17. Find the area of the region enclosed between the circles $x^2+y^2=1$ and $(x-1)^2+y^2=1$. **[CBSE 2007, '13C]**

18. Sketch the region common to the circle $x^2+y^2=16$ and the parabola $x^2=6y$. Also, find the area of the region, using integration.

19. Sketch the region common to the circle $x^2+y^2=25$ and the parabola $y^2=8x$. Also, find the area of the region, using integration.

20. Draw a rough sketch of the region $\{(x, y): y^2 \le 3x,\ 3x^2+3y^2 \le 16\}$ and find the area enclosed by the region, using the method of integration.

21. Draw a rough sketch and find the area of the region bounded by the parabolas $y^2=4x$ and $x^2=4y$, using the method of integration.

22. Find by integration the area bounded by the curve $y^2=4ax$ and the lines $y=2a$ and $x=0$.

23. Find the area between the curve $y=\frac{x}{\pi}+2\sin^2 x$, the x-axis and the ordinates $x=0$ and $x=\pi$.

24. Find the area bounded by the curve $y = \cos x$, the x-axis and the ordinates $x = 0$ and $x = 2\pi$.
25. Compare the areas under the curves $y = \cos^2 x$ and $y = \sin^2 x$ between $x = 0$ and $x = \pi$.
26. Using integration, find the area of the triangle, the equations of whose sides are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$.
27. Find the area of the region $\{(x, y) : x^2 \le y \le x\}$.
28. Find the area of the region bounded by the curve $y^2 = 2y - x$ and the y-axis.
29. Draw a rough sketch of the curves $y = \sin x$ and $y = \cos x$, as x varies from 0 to $\frac{\pi}{2}$, and find the area of the region enclosed between them and the x-axis.
30. Find the area of the region bounded by the parabola $y^2 = 2x + 1$ and the line $x - y = 1$.
31. Find the area of the region bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$.
32. Find the area of the region bounded by the curve $(y - 1)^2 = 4(x + 1)$ and the line $y = x - 1$.
33. Find the area of the region bounded by the curve $y = \sqrt{x}$ and the line $y = x$.
34. Find the area of the region included between the parabola $y^2 = 3x$ and the circle $x^2 + y^2 - 6x = 0$, lying in the first quadrant.
35. Find the area bounded by the curve $y = \cos x$ between $x = 0$ to $x = 2\pi$.
36. Using integration, find the area of the region in the first quadrant, enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. **[CBSE 2014]**
37. Using integration, find the area of the triangle whose vertices are $A(2, 3)$, $B(4, 7)$ and $C(6, 2)$. **[CBSE 2012C]**
38. Using integration, find the area of the triangle whose vertices are $A(1, 3)$, $B(2, 5)$ and $C(3, 4)$. **[CBSE 2009C]**
39. Using integration, find the area of the triangular region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. **[CBSE 2011]**

ANSWERS (EXERCISE 17)

1. $\frac{26}{3}$ sq units **2.** $\frac{28}{3}$ sq units **3.** $\frac{56}{9}$ sq units

4. 60 sq units **5.** $\frac{\pi a^2}{4}$ sq units **6.** 96 sq units

7. $8\sqrt{3}$ sq units **8.** $\frac{3\pi}{2}$ sq units **9.** $\frac{27}{2}$ sq units

10. $\frac{14}{3}$ sq units **11.** 4 sq units **12.** $\frac{15}{2}$ sq units

13. $\frac{32}{3}$ sq units **14.** 8.5 sq units **15.** $\frac{1}{3}(8 - 2\sqrt{2})$ sq units

16. $\frac{14}{9}$ sq units **17.** $\left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right)$ sq units **18.** $\frac{16\pi+4\sqrt{3}}{3}$ sq units

19. $\left\{\frac{8\sqrt{2}}{3}\cdot a^{3/2}+\frac{25\pi}{2}-a\sqrt{25-a^2}-25\sin^{-1}\frac{a}{5}\right\}$ sq units, where $a=\sqrt{41}-4$.

20. $\left\{\frac{4}{\sqrt{3}}a^{3/2}+\frac{8\pi}{3}-a\sqrt{\frac{16}{3}-a^2}-\frac{16}{3}\sin^{-1}\left(\frac{\sqrt{3}a}{4}\right)\right\}$ sq units, where $a=\frac{-9+\sqrt{273}}{6}$

21. $\frac{16}{3}$ sq units **22.** $\frac{2a^2}{3}$ sq units **23.** $\frac{3\pi}{2}$ sq units

24. 4 sq units **25.** $\frac{\pi}{2}$ sq units each **26.** 8 sq units

27. $\frac{1}{6}$ sq unit **28.** $\frac{4}{3}$ sq units **29.** $(2-\sqrt{2})$ sq units

30. $\frac{16}{3}$ sq units **31.** $\frac{9}{2}$ sq units **32.** $\frac{64}{3}$ sq units

33. $\frac{1}{6}$ sq unit **34.** $\frac{3}{4}(3\pi-8)$ sq units **35.** 4 sq units

36. 4π sq units **37.** 9 sq units **38.** $\frac{3}{2}$ sq units

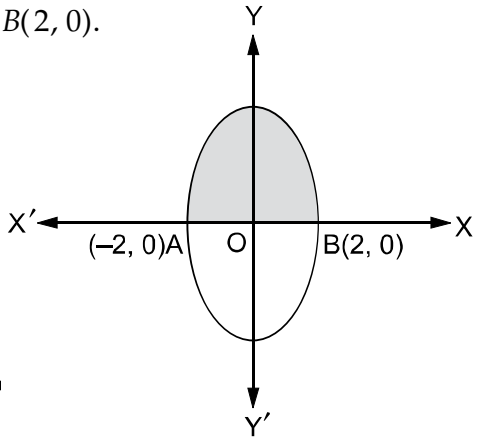
39. 8 sq units

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 17)

7. Since the equation $y^2=4x$ contains only even powers of y, the curve is symmetrical about the y-axis.

   $\therefore$ required area $=2\cdot\int_0^3 2\sqrt{x}\,dx.$

8. The given ellipse meets the x-axis at $A(-2,0)$ and $B(2,0)$.



   $\frac{x^2}{4}+\frac{y^2}{9}=1\Rightarrow y=\frac{3}{2}\cdot\sqrt{4-x^2}.$

   $\therefore$ required area $=\int_{-2}^{2} y\,dx=\frac{3}{2}\cdot\int_{-2}^{2}\sqrt{4-x^2}\,dx$

   $=2\times\frac{3}{2}\times\int_0^2\sqrt{4-x^2}\,dx$

   $=6\cdot\int_0^{\pi/2}\cos^2\theta\,d\theta$, where $x=2\sin\theta$

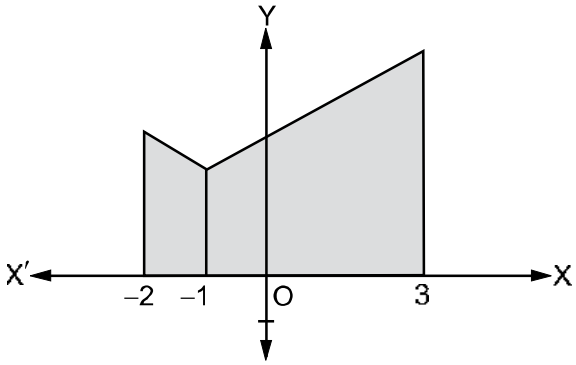
   $=3\cdot\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta.$

9. $y=1+|x+1|=\begin{cases}1+(x+1), \text{ when } x+1\geq 0\\ 1-(x+1), \text{ when } x+1<0\end{cases}$

$$= \begin{cases} x+2, \text{ when } x \geq -1 \\ -x, \text{ when } x < -1 \end{cases}$$

$$\text{Required area} = \int_{-2}^{3} y\,dx = \int_{-2}^{-1} y\,dx + \int_{-1}^{3} y\,dx$$

$$= \int_{-2}^{-1} (-x)\,dx + \int_{-1}^{3} (x+2)\,dx.$$



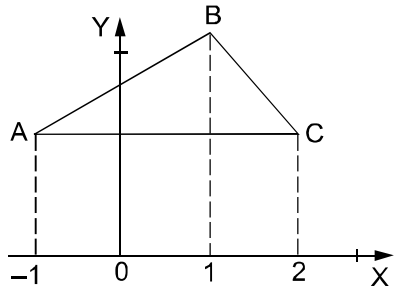
10. Required area $= \int_0^3 x\,dy = \int_0^3 \sqrt{4-y}\,dy.$

11. Equations of AB, BC and AC are

$$y = \frac{1}{2}(3x+7), y = \frac{1}{2}(11-x)$$

and $y = \frac{1}{2}(x+5)$ respectively.

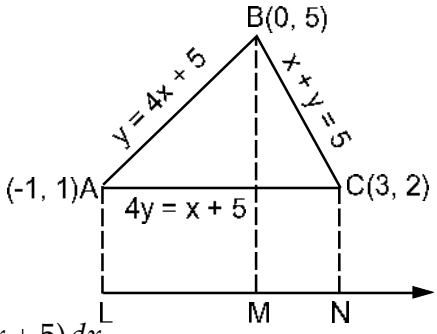
$$\therefore \quad \Delta = \frac{1}{2}\int_{-1}^{1}(3x+7)\,dx + \frac{1}{2}\int_{1}^{3}(11-x)\,dx - \frac{1}{2}\int_{-1}^{3}(x+5)\,dx.$$



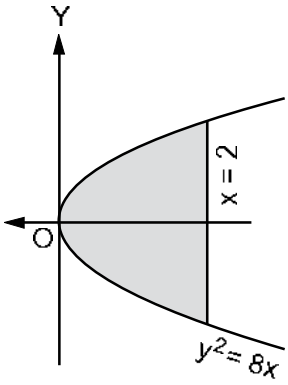
12. Solving the given equations in pairs, we get the points $A(-1, 1)$, $B(0, 5)$ and $C(3, 2)$.
Draw AL, BM and CN perpendiculars on the x-axis.

$$\text{Required area} = \int_{-1}^{0} y_{AB}\,dx + \int_{0}^{3} y_{BC}\,dx - \int_{-1}^{3} y_{CA}\,dx$$

$$= \int_{-1}^{0}(4x+5)\,dx + \int_{0}^{3}(-x+5)\,dx - \frac{1}{4}\cdot\int_{-1}^{3}(x+5)\,dx.$$



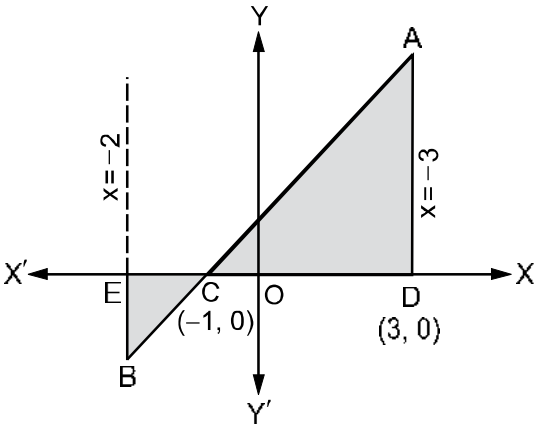
13. Required area $= 2\cdot\int_0^2 \sqrt{8x}\,dx.$



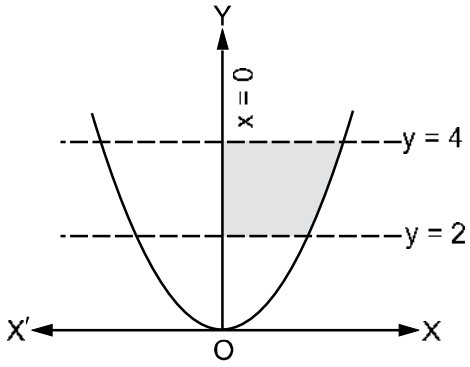
14. Let AB be the given line, intersecting the x-axis at $C(-1, 0)$.
Required area = (area $CDAC$ + area $CBEC$)

$$= \int_{-1}^{3} y\,dx + \int_{-2}^{-1}(-y)\,dx$$

$$= \int_{-1}^{3}(x+1)\,dx + \int_{-2}^{-1} -(x+1)\,dx.$$

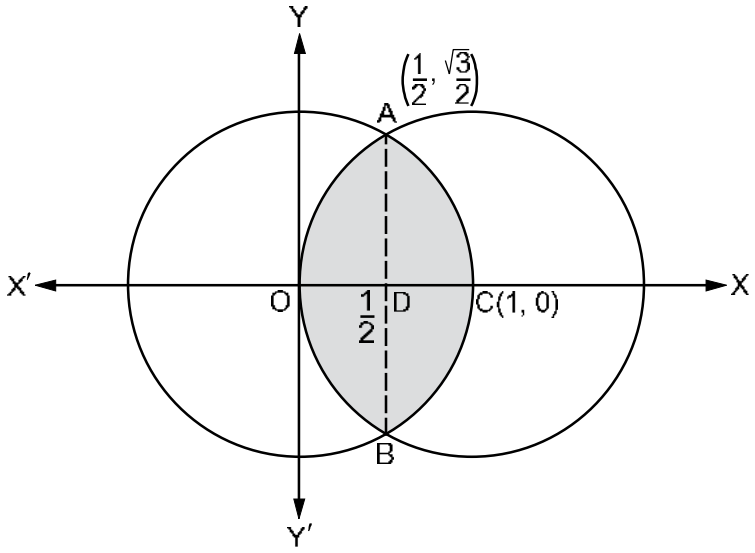


15. Clearly, we have to find the area of the region bounded by the curve $y = 4x^2$, the y-axis, and the lines $y = 2$ and $y = 4$.



Required area $= \int_2^4 x\,dy = \int_2^4 \frac{1}{2}\sqrt{y}\,dy.$

17. $(x^2 + y^2) = 1$ is a circle with its centre at (0, 0) and radius = 1 unit.



And, $(x - 1)^2 + y^2 = 1$ is a circle with its centre at (1, 0) and radius = 1 unit.

The given equations are

$$(x^2 + y^2) = 1 \qquad \ldots \text{(i)}$$

$$\text{and } (x - 1)^2 + y^2 = 1. \qquad \ldots \text{(ii)}$$

Using (i) in (ii), we get $1 - 2x = 0 \Rightarrow x = \frac{1}{2}$.

Putting $x = \frac{1}{2}$ in (i), we get $y = \pm\frac{\sqrt{3}}{2}$.

So, the two circles intersect at $A\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $B\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

Required area = area $AOBCA$

$= 2(\text{area } AOCA) = 2(\text{area } ODAO + \text{area } DCAD)$

$$= 2\left\{\int_0^{1/2} \sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1} \sqrt{1-x^2}\right\}$$

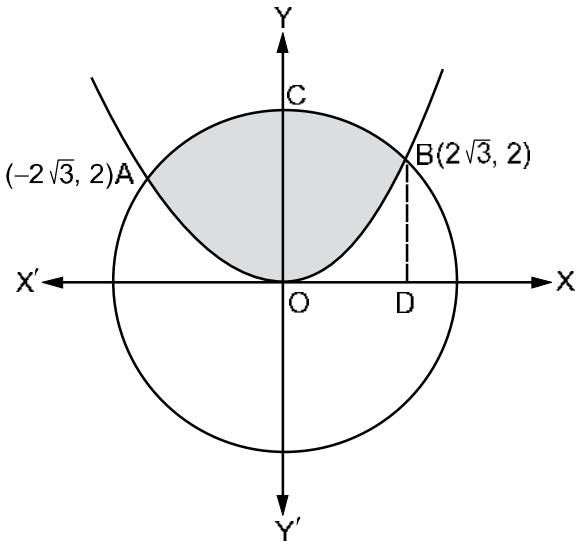
18. The given curves are

$$x^2 + y^2 = 16 \qquad \ldots \text{(i)}$$

$$\text{and} \quad x^2 = 6y \qquad \ldots \text{(ii)}$$

On solving (i) and (ii), we get the points

$A(-2\sqrt{3}, 2)$ and $B(2\sqrt{3}, 2)$.



Required area

$= 2(\text{area } OBCO)$

$= 2[\text{area } ODBCO - \text{area } ODBO]$

$$= 2\left\{\int_0^{2\sqrt{3}} \sqrt{16 - x^2}\, dx - \int_0^{2\sqrt{3}} \frac{x^2}{6} dx\right\}.$$

19. The given curves are

$$x^2 + y^2 = 25 \quad \ldots \text{(i)}$$

and $$y^2 = 8x \quad \ldots \text{(ii)}$$

On solving (i) and (ii), we get

$$x^2 + 8x - 25 = 0 \Rightarrow x = \frac{-8 \pm \sqrt{64 + 100}}{2}$$

$$\Rightarrow x = -4 + \sqrt{41} \quad \text{(rejecting –ve value).}$$

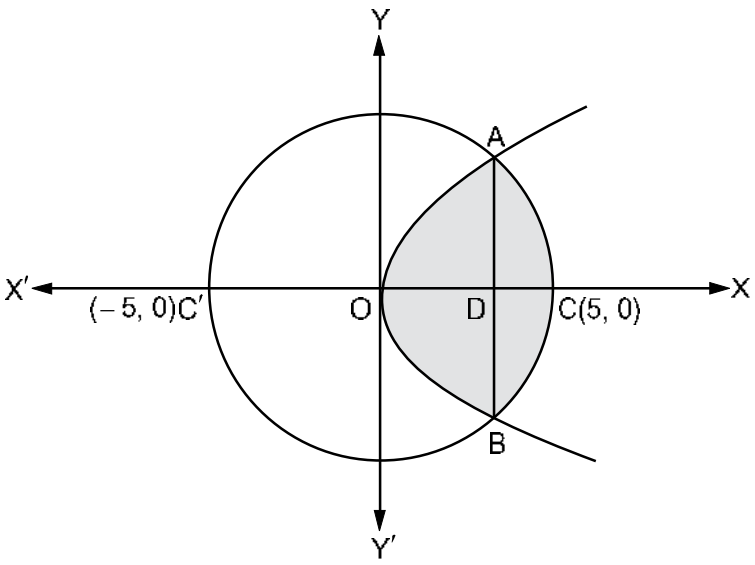
Putting $y = 0$ in (i), we get $x = \pm 5$.

Thus, the circle (i) cuts the x-axis at $C(5, 0)$ and $C'(-5, 0)$.

Required area $= 2\,[\text{area } ODCAO]$

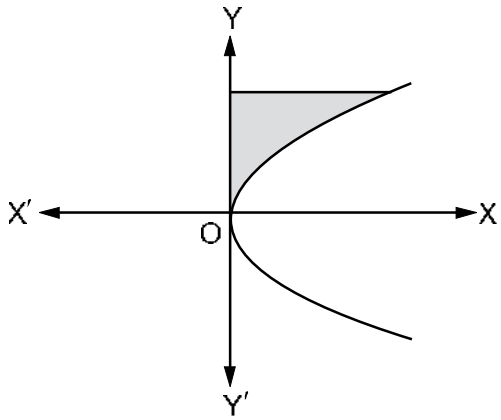
$$= 2\,[(\text{area } ODAO) + (\text{area } ADCA)]$$

$$= 2\left(\int_0^a \sqrt{8x}\, dx + \int_a^5 \sqrt{25 - x^2}\, dx\right), \text{ where } a = -4 + \sqrt{41}.$$



21. See Example 14, taking $a = 1$.

22. Required area $= \int_0^{2a} x\,dy = \int_0^{2a} \frac{y^2}{4a}\,dy = \frac{2a^2}{3}$ sq units.



28. $y^2 = 2y - x \Rightarrow y^2 - 2y + 1 = -x + 1$

$\Rightarrow (y-1)^2 = -(x-1)$.

So, the given equation represents a parabola that opens on the left, having vertex at (1, 1).

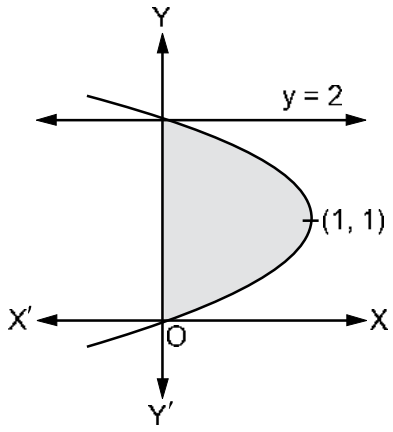
Now, $x = 0 \Rightarrow y^2 = 2y \Rightarrow y^2 - 2y = 0$

$\Rightarrow y(y-2) = 0$

$\Rightarrow y = 0$ or $y = 2$.

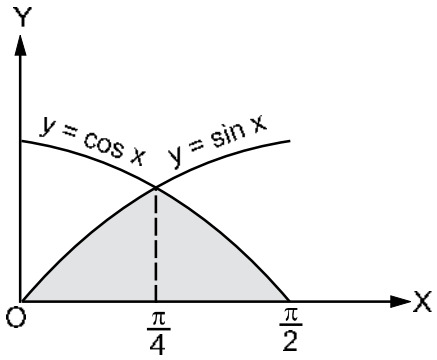
Thus, the curve meets the y-axis at (0, 0) and (0, 2)

Required area $= \int_0^2 (2y - y^2)\,dy$.



29. A rough sketch of $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$ is shown herewith.

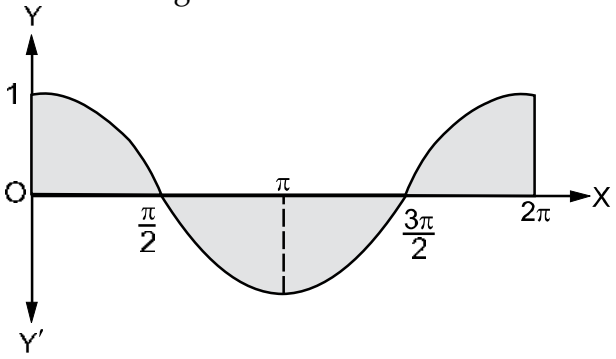
Required area $= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx$.



35. Tabular values of $y = \cos x$:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	1	0	–1	0	1

A rough sketch of the curve is given below:



Required area $= \int_0^{\pi/2} \cos x\,dx + \int_{\pi/2}^{3\pi/2} -\cos x\,dx + \int_{3\pi/2}^{2\pi} \cos x\,dx$

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36. Analogous to Example 27.

Required area $= \int_0^4 x\,dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2}\,dx$

$$= \left[\frac{x^2}{2}\right]_0^4 + \left[\frac{x\sqrt{32 - x^2}}{2} + \frac{32}{2}\sin^{-1}\frac{x}{4\sqrt{2}}\right]_4^{4\sqrt{2}}$$

$= 4\pi$ sq units.

39. $y = 2x + 1$ is the line passing through $A\left(\frac{-1}{2}, 0\right)$ and $B(0, 1)$.

$y = 3x + 1$ is the line passing through $C\left(\frac{-1}{3}, 0\right)$ and $B(0, 1)$.

$x = 4$ is the line parallel to y-axis and passing through $D(4, 0)$.

Required area $=$ (area $ODFB$) $-$ (area $ODEB$)

$$= \int_0^4 (3x + 1)\,dx - \int_0^4 (2x + 1)\,dx$$

$= (28 - 20)$ sq units $= 8$ sq units